

SPACES OF QUASI-EXPONENTIALS AND REPRESENTATIONS OF THE YANGIAN $Y(\mathfrak{gl}_N)$

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ABSTRACT. We consider a tensor product $V(\mathbf{b}) = \otimes_{i=1}^n \mathbb{C}^N(b_i)$ of the Yangian $Y(\mathfrak{gl}_N)$ evaluation vector representations. We consider the action of the commutative Bethe subalgebra $\mathcal{B}^q \subset Y(\mathfrak{gl}_N)$ on a \mathfrak{gl}_N -weight subspace $V(\mathbf{b})_\lambda \subset V(\mathbf{b})$ of weight λ . Here the Bethe algebra depends on the parameters $\mathbf{q} = (q_1, \dots, q_N)$. We identify the \mathcal{B}^q -module $V(\mathbf{b})_\lambda$ with the regular representation of the algebra of functions on a fiber of a suitable discrete Wronski map. If $\mathbf{q} = (1, \dots, 1)$, we study the action of $\mathcal{B}^{\mathbf{q}=1}$ on a space $V(\mathbf{b})_\lambda^{\text{sing}}$ of singular vectors of a certain weight. Again, we identify the $\mathcal{B}^{\mathbf{q}=1}$ -module $V(\mathbf{b})_\lambda^{\text{sing}}$ with the regular representation of the algebra of functions on a fiber of another suitable discrete Wronski map.

These results we announced earlier in relation with a description of the quantum equivariant cohomology of the cotangent bundle of a partial flag variety and a description of commutative subalgebras of the group algebra of a symmetric group.

1. INTRODUCTION

A Bethe algebra of a quantum integrable model is a commutative algebra of linear operators (Hamiltonians) acting on the vector space of states of the model. An interesting problem is to describe the Bethe algebra as the algebra of functions on a suitable scheme. Such a description can be considered as an instance of the geometric Langlands correspondence, see for example [MTV4]. The \mathfrak{gl}_N XXX model is an example of a quantum integrable model. The Bethe algebra \mathcal{B}^q of the XXX model is a commutative subalgebra of the Yangian $Y(\mathfrak{gl}_N)$. The algebra \mathcal{B}^q depends on the parameters $q = (q_1, \dots, q_N) \in \mathbb{C}^N$. Having a $Y(\mathfrak{gl}_N)$ -module M , one obtains the commutative subalgebra $\mathcal{B}^q(M) \subset \text{End}(M)$ as the image of \mathcal{B}^q . The geometric interpretation of the algebra $\mathcal{B}^q(M)$ as the algebra of functions on a scheme leads to interesting objects, see for example, [GRTV].

In this paper, we consider (among other Yangian modules) a tensor product $V(\mathbf{b}) = \otimes_{i=1}^n \mathbb{C}^N(b_i)$ of the Yangian $Y(\mathfrak{gl}_N)$ evaluation vector representations. We consider the action of the Bethe subalgebra $\mathcal{B}^q \subset Y(\mathfrak{gl}_N)$ on a \mathfrak{gl}_N -weight subspace $V(\mathbf{b})_\lambda \subset V(\mathbf{b})$ of weight λ . We identify the \mathcal{B}^q -module $V(\mathbf{b})_\lambda$ with the regular representation of the algebra of functions

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on a fiber of a suitable discrete Wronski map. If $\mathbf{q} = (1, \dots, 1)$, we study the action of $\mathcal{B}^{\mathbf{q}=1}$ on a space $V(\mathbf{b})_{\lambda}^{\text{sing}}$ of singular vectors of a certain weight. Again, we identify the $\mathcal{B}^{\mathbf{q}=1}$ -module $V(\mathbf{b})_{\lambda}^{\text{sing}}$ with the regular representation of the algebra of functions on a fiber of another suitable discrete Wronski map.

These results are parallel to the analogous results of [MTV3, MTV4], where we study the corresponding $\mathfrak{gl}_N[t]$ -modules instead of the Yangian $Y(\mathfrak{gl}_N)$ -modules.

We used the results of this paper earlier in [GRTV, Theorems 6.3-6.5] in relation with a description of the quantum equivariant cohomology of the cotangent bundle of a partial flag variety and in [MTV6, Theorem 7.3] in relation with a description of commutative subalgebras of the group algebra of a symmetric group. More details are given in remarks after Theorem 5.2 and at the end of Section 6.

In Section 2, we consider the space $\mathcal{V} = (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[z_1, \dots, z_n]$, an action on \mathcal{V} of the symmetric group S_n , the subspace $\mathcal{V}^S \subset \mathcal{V}$ of the S_n -invariants, the \mathfrak{gl}_N weight subspaces $(\mathcal{V}^S)_{\lambda} \subset \mathcal{V}^S$ and the subspaces $(\mathcal{V}^S)_{\lambda}^{\text{sing}} \subset (\mathcal{V}^S)_{\lambda}$ of singular vectors.

In Section 3, we introduce an action of the Yangian $Y(\mathfrak{gl}_N)$ on \mathcal{V}^S . In Section 4, we introduce Bethe subalgebras $\mathcal{B}^{\mathbf{q}} \subset Y(\mathfrak{gl}_N)$. The induced $\mathcal{B}^{\mathbf{q}}$ -action on \mathcal{V}^S preserves the weight subspaces $(\mathcal{V}^S)_{\lambda}$. If $\mathbf{q} = (1, \dots, 1)$, then the $\mathcal{B}^{\mathbf{q}=1}$ -action on \mathcal{V}^S preserves the subspaces $(\mathcal{V}^S)_{\lambda}^{\text{sing}}$ of singular vectors.

In Section 5, we introduce a discrete Wronski map on collections of quasi-exponentials. Theorem 5.2 describes the $\mathcal{B}^{\mathbf{q}}$ -module $(\mathcal{V}^S)_{\lambda}$ for \mathbf{q} with distinct coordinates in terms of the discrete Wronski map. In Section 6 we define a discrete Wronski map on collections of polynomials. Theorem 6.3 describes the $\mathcal{B}^{\mathbf{q}=1}$ -module $(\mathcal{V}^S)_{\lambda}^{\text{sing}}$ in terms of the second Wronski map. Corollaries 5.4 and 6.4 give an application of Theorems 5.2 and 6.3 to a description of the Bethe algebra action on a tensor product of evaluation vector representations.

Proofs of the theorems are based of the Bethe ansatz. We prove the corresponding Bethe ansatz statements in Sections 7 and 8, and prove Theorems 5.2 and 6.3 in Section 9.

In Section 10, we consider the S_n -skew-invariant part $\mathcal{V}^A \subset \mathcal{V}$ and the space $\frac{1}{D}\mathcal{V}^A$ of S_n -invariant rational functions. Theorems 10.6 and 10.8 describe the $\mathcal{B}^{\mathbf{q}}$ -module $(\frac{1}{D}\mathcal{V}^A)_{\lambda}$ for \mathbf{q} with distinct coordinates and the $\mathcal{B}^{\mathbf{q}=1}$ -module $(\frac{1}{D}\mathcal{V}^A)_{\lambda}^{\text{sing}}$ in terms of the corresponding Wronski maps.

2. SPACE \mathcal{V}^S

2.1. Lie algebra \mathfrak{gl}_N . Let $e_{i,j}$, $i, j = 1, \dots, N$, be the standard generators of the Lie algebra \mathfrak{gl}_N satisfying the relations $[e_{i,j}, e_{k,l}] = \delta_{j,k}e_{i,l} - \delta_{i,l}e_{k,j}$. We denote by $\mathfrak{h} \subset \mathfrak{gl}_N$ the subalgebra generated by $e_{i,i}$, $i = 1, \dots, N$. For a Lie algebra \mathfrak{g} , we denote by $U(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} .

A vector v of a \mathfrak{gl}_N -module M has weight $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ if $e_{i,i}v = \lambda_i v$ for $i = 1, \dots, N$. A vector v is singular if $e_{i,j}v = 0$ for $1 \leq i < j \leq N$.

We denote by M_{λ} the subspace of M of weight λ , by M^{sing} the subspace of M of all singular vectors and by $M_{\lambda}^{\text{sing}}$ the subspace of M of all singular vectors of weight λ .

A sequence of integers $\lambda = (\lambda_1, \dots, \lambda_N)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ is called a partition with at most N parts. Set $|\lambda| = \sum_{i=1}^N \lambda_i$. We say that λ is a partition of $|\lambda|$.

Let \mathbb{C}^N be the standard vector representation of \mathfrak{gl}_N with basis v_1, \dots, v_N such that $e_{i,j}v_k = \delta_{j,k}v_i$ for all i, j, k . A tensor power $V = (\mathbb{C}^N)^{\otimes n}$ of the vector representation has a basis given by the vectors $v_{i_1} \otimes \dots \otimes v_{i_n}$, where $i_j \in \{1, \dots, N\}$. Every such sequence (i_1, \dots, i_n) defines a decomposition $I = (I_1, \dots, I_N)$ of $\{1, \dots, n\}$ into disjoint subsets I_1, \dots, I_N , where $I_j = \{k \mid i_k = j\}$. We denote the basis vector $v_{i_1} \otimes \dots \otimes v_{i_n}$ by v_I .

Let

$$V = \bigoplus_{\lambda \in \mathbb{Z}_{\geq 0}^N, |\lambda|=n} V_\lambda$$

be the weight decomposition. Denote \mathcal{I}_λ the set of all indices I with $|I_j| = \lambda_j$, $j = 1, \dots, N$. The vectors $\{v_I \mid I \in \mathcal{I}_\lambda\}$ form a basis of V_λ .

2.2. Space \mathcal{V}^S . Let \mathcal{V} be the space of polynomials in $\mathbf{z} = (z_1, \dots, z_n)$ with coefficients in $V = (\mathbb{C}^N)^{\otimes n}$:

$$\mathcal{V} = V \otimes_{\mathbb{C}} \mathbb{C}[z_1, \dots, z_n].$$

We embed the space V into \mathcal{V} by sending $v \in V$ to $v \otimes 1 \in \mathcal{V}$.

Consider the grading on $\mathbb{C}[z_1, \dots, z_n]$, $\deg z_i = 1$ for $i = 1, \dots, n$. We define the degree of elements of \mathcal{V} by the rule $\deg(v \otimes p) = \deg p$. We consider the increasing filtration $F_0\mathcal{V} \subset F_1\mathcal{V} \subset \dots \subset \mathcal{V}$ whose k -th subspace consists of elements of degree $\leq k$. The filtration on \mathcal{V} induces a natural filtration on $\text{End}(\mathcal{V})$.

Let $P^{(i,j)}$ be the permutation of the i -th and j -th factors of $V = (\mathbb{C}^N)^{\otimes n}$. Let $s_1, \dots, s_{n-1} \in S_n$ be the elementary transpositions. We define an S_n -action on V -valued functions of z_1, \dots, z_n by the formula:

$$(2.1) \quad s_i : f(z_1, \dots, z_n) \mapsto \frac{(z_i - z_{i+1}) P^{(i,i+1)} - 1}{z_i - z_{i+1}} f(z_1, \dots, z_{i+1}, z_i, \dots, z_n) + \\ + \frac{1}{z_i - z_{i+1}} f(z_1, \dots, z_i, z_{i+1}, \dots, z_n).$$

These formulae induce an S_n -action on \mathcal{V} . The S_n -action preserves the filtration: for any k we have $S_n \times F_k\mathcal{V} \rightarrow F_k\mathcal{V}$. We denote by \mathcal{V}^S the subspace of S_n -invariants in \mathcal{V} .

The group S_n acts on the algebra $\mathbb{C}[z_1, \dots, z_n]$ by permuting the variables. Let $\sigma_i(\mathbf{z})$, $i = 1, \dots, n$, be the s -th elementary symmetric polynomial in z_1, \dots, z_n . The algebra of symmetric polynomials $\mathbb{C}[z_1, \dots, z_n]^S$ is a free polynomial algebra with generators $\sigma_1(\mathbf{z}), \dots, \sigma_n(\mathbf{z})$.

Lemma 2.1. *The space \mathcal{V}^S is a free $\mathbb{C}[z_1, \dots, z_n]^S$ -module of rank N^n .*

Proof. The lemma follows from Lemma 2.10 in [GRTV]. □

The Lie algebra \mathfrak{gl}_N naturally acts on \mathcal{V} preserving the grading and commuting with the S_n -action on \mathcal{V} . Therefore, \mathcal{V}^S is a filtered \mathfrak{gl}_N -module. We consider the \mathfrak{gl}_N -weight decomposition

$$\mathcal{V}^S = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0}^N \\ |\lambda|=n}} (\mathcal{V}^S)_\lambda,$$

as well as the subspaces of singular vectors $(\mathcal{V}^S)_\lambda^{\text{sing}} \subset (\mathcal{V}^S)_\lambda$. All of these are filtered free $\mathbb{C}[z_1, \dots, z_n]^S$ -modules.

Let M be a $\mathbb{Z}_{\geq 0}$ -filtered space with finite-dimensional graded components $F_k M / F_{k-1} M$. We call the formal power series in a variable t ,

$$\text{ch}_M(t) = \sum_{k=0}^{\infty} (\dim F_k M / F_{k-1} M) t^k,$$

the graded character of M . We set $(t)_a = \prod_{j=1}^a (1 - t^j)$.

Lemma 2.2. *For $\lambda \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$, we have*

$$(2.2) \quad \text{ch}_{(\mathcal{V}^S)_\lambda}(t) = \prod_{i=1}^N \frac{1}{(t)_{\lambda_i}}.$$

For a partition λ of n with at most N parts, we have

$$(2.3) \quad \text{ch}_{(\mathcal{V}^S)_\lambda^{\text{sing}}}(t) = \frac{\prod_{1 \leq i < j \leq N} (1 - t^{\lambda_i - \lambda_j + j - i})}{\prod_{i=1}^N (t)_{\lambda_i + N - i}} t^{\sum_{i=1}^N (i-1)\lambda_i}.$$

Proof. The S_n -action on the graded components $F_k \mathcal{V}_\lambda / F_{k-1} \mathcal{V}_\lambda$ and $F_k \mathcal{V}_\lambda^{\text{sing}} / F_{k-1} \mathcal{V}_\lambda^{\text{sing}}$ coincides with the S_n -action considered in [MTV2] and [MTV3]. Formula (2.2) follows from [MTV3, Lemma 2.12]. Formula (2.3) follows from [MTV2, Formula 5.3] and [MTV2, Lemma 2.2]. \square

Given $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$, denote by $I_{\mathbf{a}} \subset \mathbb{C}[z_1, \dots, z_n]^S$ the ideal generated by the polynomials $\sigma_i(\mathbf{z}) - a_i$, $i = 1, \dots, n$. For any \mathbf{a} , the quotient $\mathcal{V}^S / I_{\mathbf{a}} \mathcal{V}^S$ is a complex vector space of dimension N^n by Lemma 2.1.

3. YANGIAN MODULES

3.1. Yangian $Y(\mathfrak{gl}_N)$. The Yangian $Y(\mathfrak{gl}_N)$ is the unital associative algebra with generators $T_{i,j}^{\{s\}}$ for $i, j = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, subject to relations

$$(3.1) \quad (u - v) [T_{i,j}(u), T_{k,l}(v)] = T_{k,j}(v) T_{i,l}(u) - T_{k,j}(u) T_{i,l}(v), \quad i, j, k, l = 1, \dots, N,$$

where

$$T_{i,j}(u) = \delta_{i,j} + \sum_{s=1}^{\infty} T_{i,j}^{\{s\}} u^{-s}.$$

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct $\Delta : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ given by

$$\Delta : T_{i,j}(u) \mapsto \sum_{k=1}^N T_{k,j}(u) \otimes T_{i,k}(u)$$

for $i, j = 1, \dots, N$. The Yangian $Y(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a Hopf subalgebra, the embedding given by $e_{i,j} \mapsto T_{j,i}^{\{1\}}$.

The Yangian $Y(\mathfrak{gl}_N)$ has the degree function such that $\deg T_{i,j}^{\{s\}} = s - 1$ for any $i, j = 1, \dots, N$, $s = 1, 2, \dots$. The Yangian $Y(\mathfrak{gl}_N)$ is a filtered algebra with the increasing filtration $F_0 Y(\mathfrak{gl}_N) \subset F_1 Y(\mathfrak{gl}_N) \subset \dots \subset Y(\mathfrak{gl}_N)$, where $F_s Y(\mathfrak{gl}_N)$ consists of elements of degree $\leq s$.

There is a one-parameter family of automorphisms

$$\rho_b : Y(\mathfrak{gl}_N) \rightarrow Y(\mathfrak{gl}_N), \quad T_{i,j}(u) \mapsto T_{i,j}(u - b),$$

where $b \in \mathbb{C}$ and $(u - b)^{-1}$ in the right-hand side has to be expanded as a power series in u^{-1} .

The evaluation homomorphism $\epsilon : Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N)$ is defined by the rule: $T_{i,j}^{\{1\}} \mapsto e_{j,i}$ for all i, j , and $T_{i,j}^{\{s\}} \mapsto 0$ for all i, j and all $s > 1$.

For a \mathfrak{gl}_N -module M and $b \in \mathbb{C}$, we denote by $M(b)$ the $Y(\mathfrak{gl}_N)$ -module induced from M by the homomorphism $\epsilon \cdot \rho_b$. We call it the evaluation module with the evaluation point b .

Recall that we consider \mathbb{C}^N as the \mathfrak{gl}_N -module with highest weight $(1, 0, \dots, 0)$. For any $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$, we obtain the $Y(\mathfrak{gl}_N)$ -module

$$V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n).$$

3.2. Yangian module \mathcal{V}^S . Consider $\mathbb{C}^N \otimes V = \mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes n}$, where the factors are labeled by $0, 1, \dots, n$. Set

$$L(u) = (u - z_n + P^{(0,n)}) \dots (u - z_1 + P^{(0,1)}).$$

This is a polynomial in u, z_1, \dots, z_n with values in $\text{End}(\mathbb{C}^N \otimes V)$. We consider $L(u)$ as an $N \times N$ matrix with $\text{End}(V) \otimes \mathbb{C}[u, z_1, \dots, z_n]$ -valued entries $L_{i,j}(u)$, $i, j = 1, \dots, N$.

Lemma 3.1. *The assignment*

$$(3.2) \quad \phi : T_{i,j}(u) \mapsto L_{i,j}(u) \prod_{a=1}^n (u - z_a)^{-1}$$

defines the $Y(\mathfrak{gl}_N)$ -module structure on $\mathcal{V} = (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[z_1, \dots, z_n]$. We consider the right-hand side of (3.2) as a series in u^{-1} with coefficients in $\text{End}(V) \otimes \mathbb{C}[z_1, \dots, z_n]$.

Proof. The Yang-Baxter equation

$$(3.3) \quad \begin{aligned} (u - v + hP^{(1,2)})(u + hP^{(1,3)})(v + hP^{(2,3)}) &= \\ &= (v + hP^{(2,3)})(u + hP^{(1,3)})(u - v + hP^{(1,2)}). \end{aligned}$$

implies that

$$(u - v + P^{(1,2)}) L^{(1)}(u) L^{(2)}(v) = L^{(2)}(v) L^{(1)}(u) (u - v + P^{(1,2)}),$$

which means

$$(u - v) [L_{i,j}(u), L_{k,l}(v)] = L_{k,j}(v) L_{i,l}(u) - L_{k,j}(u) L_{i,l}(v)$$

for all $i, j, k, l = 1, \dots, N$. Comparing the last formula with the defining relations (3.1) for the Yangian $Y(\mathfrak{gl}_N)$ completes the proof. \square

The subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$ acts on \mathcal{V} in the standard way: an element $x \in \mathfrak{gl}_N$ acts as $x^{(1)} + \dots + x^{(n)}$.

Lemma 3.2. *The $Y(\mathfrak{gl}_N)$ -action on \mathcal{V} is filtered: for any k, s , we have $F_s Y(\mathfrak{gl}_N) \times F_k \mathcal{V} \rightarrow F_{s+k} \mathcal{V}$. \square*

Lemma 3.3. *The $Y(\mathfrak{gl}_N)$ -action ϕ on \mathcal{V} commutes with the S_n -action (2.1) and with multiplication by the elements of $\mathbb{C}[z_1, \dots, z_n]$.*

Proof. The first part follows from the Yang-Baxter equation (3.3), and the second part is clear. \square

By Lemma 3.3, the action ϕ makes the space \mathcal{V}^S into a filtered $Y(\mathfrak{gl}_N)$ -module. For any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{C}^N$, the subspace $I_{\mathbf{a}}\mathcal{V}^S$ is a $Y(\mathfrak{gl}_N)$ -submodule.

Lemma 3.4 ([GRTV, Proposition 4.6]). *The $Y(\mathfrak{gl}_N)$ -module \mathcal{V}^S is generated by the vector $v_1^{\otimes n} = v_1 \otimes \dots \otimes v_1$.* \square

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{C}^N$, introduce complex numbers b_1, \dots, b_n by the relation

$$(3.4) \quad \prod_{s=1}^n (u - b_s) = u^n + \sum_{j=1}^n (-1)^j a_j u^{n-j}.$$

The numbers are defined up to a permutation.

Proposition 3.5. *Assume that the numbers b_1, \dots, b_n are ordered such that $b_i \neq b_j + 1$ for $i > j$. Then the $Y(\mathfrak{gl}_N)$ -module $\mathcal{V}^S/I_{\mathbf{a}}\mathcal{V}^S$ is isomorphic to $V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n)$, the tensor product of evaluation $Y(\mathfrak{gl}_N)$ -modules.*

Proof. Consider the map $\varphi : \mathcal{V}^S \rightarrow V(\mathbf{b})$ that sends every element of \mathcal{V}^S to its value at the point $\mathbf{z} = (b_1, \dots, b_n)$. This map is a homomorphism of $Y(\mathfrak{gl}_N)$ -modules and factors through the canonical projection $\vartheta : \mathcal{V}^S \rightarrow \mathcal{V}^S/I_{\mathbf{a}}\mathcal{V}^S$. Since ϑ is also a homomorphism of $Y(\mathfrak{gl}_N)$ -modules, this defines a homomorphism of $Y(\mathfrak{gl}_N)$ -modules $\psi : \mathcal{V}^S/I_{\mathbf{a}}\mathcal{V}^S \rightarrow V(\mathbf{b})$.

Under the assumption that $b_i \neq b_j + 1$ for $i > j$, the $Y(\mathfrak{gl}_N)$ -module $V(\mathbf{b})$ is generated by the vector $v_1^{\otimes n}$, see [NT2, Proposition 3.1]. Therefore, the map ψ is surjective because $\psi(v_1^{\otimes n}) = v_1^{\otimes n}$, and since $\dim \mathcal{V}^S/I_{\mathbf{a}}\mathcal{V}^S = N^n = \dim V(\mathbf{b})$, the map ψ is an isomorphism of $Y(\mathfrak{gl}_N)$ -modules. \square

Proposition 3.6. *The $Y(\mathfrak{gl}_N)$ -module $V(\mathbf{b})$ is irreducible if and only if $b_i \neq b_j + 1$ for all $i \neq j$.*

Proof. The statement follows, for instance, from [NT2, Theorem 3.4]. \square

4. BETHE SUBALGEBRAS

4.1. Bethe subalgebras. For $k = 1, \dots, N$, $\mathbf{i} = \{1 \leq i_1 < \dots < i_k \leq N\}$, $\mathbf{j} = \{1 \leq j_1 < \dots < j_k \leq N\}$, define

$$M_{\mathbf{i}, \mathbf{j}}(u) = \sum_{\sigma \in S_k} (-1)^\sigma T_{i_1, j_{\sigma(1)}}(u) \dots T_{i_k, j_{\sigma(k)}}(u - k + 1).$$

For $\mathbf{i} = \{1, \dots, N\}$, the series $M_{\mathbf{i}, \mathbf{i}}(u)$ is called the quantum determinant and denoted by $\text{qdet } T(u)$. Its coefficients generate the center of the Yangian $Y(\mathfrak{gl}_N)$.

For $\mathbf{q} = (q_1, \dots, q_N) \in (\mathbb{C}^*)^N$ and $k = 1, \dots, N$, we define

$$(4.1) \quad B_k^{\mathbf{q}}(u) = \sum_{\mathbf{i} = \{1 \leq i_1 < \dots < i_k \leq N\}} q_{i_1} \dots q_{i_k} M_{\mathbf{i}, \mathbf{i}}(u) = \sigma_k(q_1, \dots, q_N) + \sum_{s=1}^{\infty} B_{k,s}^{\mathbf{q}} u^{-s},$$

where σ_k is the k -th elementary symmetric function and $B_{k,s}^{\mathbf{q}} \in Y(\mathfrak{gl}_N)$. In particular,

$$B_N^{\mathbf{q}}(u) = q_1 \dots q_N M_{\mathbf{i}, \mathbf{i}}(u),$$

where $\mathbf{i} = \{1, \dots, N\}$. The generating series $B_k^{\mathbf{q}}(u)$, $k = 1, \dots, N$, are called the transfer-matrices.

Lemma 4.1. *We have $B_{k,s}^{\mathbf{q}} \in F_s Y(\mathfrak{gl}_N)$ for all k, s .* \square

Let $\mathcal{B}^{\mathbf{q}} \subset Y(\mathfrak{gl}_N)$ be the unital subalgebra generated by the elements $B_{k,s}^{\mathbf{q}}$, $k = 1, \dots, N$, $s > 0$. The subalgebra $\mathcal{B}^{\mathbf{q}}$ is called a Bethe subalgebra of $Y(\mathfrak{gl}_N)$. The subalgebra $\mathcal{B}^{\mathbf{q}}$ does not change if all q_1, \dots, q_N are multiplied by the same number. If $\mathbf{q} = (1, \dots, 1)$, then the corresponding Bethe subalgebra will be denoted by $\mathcal{B}^{\mathbf{q}=1}$.

Theorem 4.2 ([KS]). *The subalgebra $\mathcal{B}^{\mathbf{q}}$ is commutative and commutes with the subalgebra $U(\mathfrak{h}) \subset Y(\mathfrak{gl}_N)$. The subalgebra $\mathcal{B}^{\mathbf{q}=1}$ commutes with the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$.* \square

As a subalgebra of $Y(\mathfrak{gl}_N)$, the Bethe algebra $\mathcal{B}^{\mathbf{q}}$ acts on any $Y(\mathfrak{gl}_N)$ -module M . Since $\mathcal{B}^{\mathbf{q}}$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces M_{λ} . The subalgebra $\mathcal{B}^{\mathbf{q}=1}$ preserves the singular weight subspaces $M_{\lambda}^{\text{sing}}$.

If $L \subset M$ is a $\mathcal{B}^{\mathbf{q}}$ -invariant subspace, then the image of $\mathcal{B}^{\mathbf{q}}$ in $\text{End}(L)$ will be called the Bethe algebra of L and denoted by $\mathcal{B}^{\mathbf{q}}(L)$.

We will study the action of $\mathcal{B}^{\mathbf{q}}$ on the weight subspaces $(\mathcal{V}^S)_{\lambda}$ and the action of $\mathcal{B}^{\mathbf{q}=1}$ on the singular weight subspaces $(\mathcal{V}^S)_{\lambda}^{\text{sing}}$. The image of $B_{k,s}^{\mathbf{q}}$ in $\text{End}((\mathcal{V}^S)_{\lambda})$ will be denoted by $B_{k,s}^{\mathbf{q}, \lambda}$.

Lemma 4.3. *The generating series $B_N^{\mathbf{q}}(u)$ acts on the $Y(\mathfrak{gl}_N)$ -module \mathcal{V} as multiplication by the scalar function*

$$(4.2) \quad q_1 \dots q_N \prod_{i=1}^n \frac{u - z_i + 1}{u - z_i}.$$

\square

Corollary 4.4. *The Bethe algebra $\mathcal{B}^{\mathbf{q}}(\mathcal{V})$ contains the algebra of scalar operators of multiplication by elements of $\mathbb{C}[z_1, \dots, z_n]^S$.* \square

4.2. Universal difference operator. Define the operator τ acting on functions of u as $(\tau f)(u) = f(u - 1)$. Following [T], for $\mathbf{q} = (q_1, \dots, q_N) \in (\mathbb{C}^{\times})^N$ we introduce the universal difference operator $\mathcal{D}^{\mathbf{q}}(u, \tau)$ by the formula

$$(4.3) \quad \mathcal{D}^{\mathbf{q}}(u, \tau) = 1 + \sum_{k=1}^N (-1)^k B_k^{\mathbf{q}}(u) \tau^k.$$

For $\mathbf{q} = \mathbf{1}$, we write

$$(4.4) \quad \mathcal{D}^{\mathbf{q}=1}(u, \tau) \tau^{-N} = \sum_{k=0}^N (-1)^k C_k(u) (\tau^{-1} - 1)^{N-k}, \quad C_k(u) = \sum_{s=0}^{\infty} C_{k,s} u^{-s},$$

where $C_{k,s} \in \mathcal{B}^{q=1}$. The Bethe algebra $\mathcal{B}^{q=1}$ preserves $(\mathcal{V}^S)_\lambda^{sing}$ and we may consider the images $C_{k,s}^\lambda$ of the elements $C_{k,s}$ in $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{sing})$.

Theorem 4.5 ([MTV2, Theorem 3.7]). *The following statements hold.*

- (i) $C_0(u) = 1$.
- (ii) $C_{k,s} = 0$ for all $k = 1, \dots, N$ and $s < k$.
- (iii) $C_{1,1}^\lambda, \dots, C_{N,N}^\lambda$ are scalar operators, and for a variable x , we have

$$\sum_{k=0}^N (-1)^k C_{k,k}^\lambda \prod_{j=0}^{N-k-1} (x - j) = \prod_{s=1}^N (x - \lambda_s - N + s).$$

□

Remark. Given an $N \times N$ matrix A with possibly noncommuting entries $a_{i,j}$, we define its row determinant to be

$$\text{rdet } A = \sum_{\sigma \in S_N} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{N,\sigma(N)}.$$

The universal difference operator can be presented as a row determinant of a suitable matrix, see for example [T, MTV1, MTV2].

5. SPACES OF QUASI-EXPONENTIALS

5.1. Spaces of quasi-exponentials. Let $\mathbf{q} = (q_1, \dots, q_N) \in (\mathbb{C}^\times)^N$ be a sequence of distinct numbers. Let $\lambda \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$. Let $\Omega_\lambda^{\mathbf{q}}$ be the affine n -dimensional space with coordinates $p_{i,j}$, $i = 1, \dots, N$, $j = 1, \dots, \lambda_i$.

Introduce $f_i(u) = q_i^u p_i(u)$, $i = 1, \dots, N$, where

$$(5.1) \quad p_i(u) = u^{\lambda_i} + p_{i,1} u^{\lambda_i-1} + \dots + p_{i,\lambda_i}.$$

We identify points $X \in \Omega_\lambda^{\mathbf{q}}$ with N -dimensional complex vector spaces generated by quasi-exponentials

$$(5.2) \quad f_i(u, X) = q_i^u (u^{\lambda_i} + p_{i,1}(X) u^{\lambda_i-1} + \dots + p_{i,\lambda_i}(X)), \quad i = 1, \dots, N.$$

Denote by $\mathcal{O}_\lambda^{\mathbf{q}}$ the algebra of regular functions on $\Omega_\lambda^{\mathbf{q}}$. It is the polynomial algebra in the variables $p_{i,j}$. The algebra $\mathcal{O}_\lambda^{\mathbf{q}}$ has the degree function such that $\deg p_{i,j} = j$ for all i, j . We consider the increasing filtration $F_0 \mathcal{O}_\lambda^{\mathbf{q}} \subset F_1 \mathcal{O}_\lambda^{\mathbf{q}} \subset \dots \subset \mathcal{O}_\lambda^{\mathbf{q}}$, where $F_s \mathcal{O}_\lambda^{\mathbf{q}}$ consists of elements of degree $\leq s$. The graded character of $\mathcal{O}_\lambda^{\mathbf{q}}$ is

$$\text{ch}_{\mathcal{O}_\lambda^{\mathbf{q}}}(t) = \prod_{i=1}^N \frac{1}{(t)_{\lambda_i}}.$$

5.2. Another realization of $\mathcal{O}_\lambda^{\mathbf{q}}$. For arbitrary functions $g_1(u), \dots, g_N(u)$, we introduce the discrete Wronskian by the formula

$$\text{Wr}(g_1(u), \dots, g_N(u)) = \det \begin{pmatrix} g_1(u) & g_1(u-1) & \dots & g_1(u-N+1) \\ g_2(u) & g_2(u-1) & \dots & g_2(u-N+1) \\ \dots & \dots & \dots & \dots \\ g_N(u) & g_N(u-1) & \dots & g_N(u-N+1) \end{pmatrix}.$$

Let $f_i(u)$, $i = 1, \dots, N$, be the functions given by (5.1). We have

$$(5.3) \quad \text{Wr}(f_1(u-1), \dots, f_N(u-1)) = \prod_{i=1}^N q_i^{u-1} \prod_{1 \leq i < j \leq N} (q_j^{-1} - q_i^{-1}) \left(u^n + \sum_{s=1}^n (-1)^s \Sigma_s u^{n-s} \right),$$

where $\Sigma_1, \dots, \Sigma_n$ are elements of \mathcal{O}_λ^q . Define the difference operator $\mathcal{D}^{\mathcal{O}_\lambda^q}(u, \tau)$ by

$$(5.4) \quad \mathcal{D}^{\mathcal{O}_\lambda^q}(u, \tau) = \frac{1}{\text{Wr}(f_1(u-1), \dots, f_N(u-1))} \text{rdet} \begin{pmatrix} f_1(u) & f_1(u-1) & \dots & f_1(u-N) \\ f_2(u) & f_2(u-1) & \dots & f_2(u-N) \\ \dots & \dots & \dots & \dots \\ 1 & \tau & \dots & \tau^N \end{pmatrix}.$$

It is a difference operator in the variable u , whose coefficients are formal power series in u^{-1} with coefficients in \mathcal{O}_λ^q ,

$$(5.5) \quad \mathcal{D}^{\mathcal{O}_\lambda^q}(u, \tau) = 1 + \sum_{k=1}^N (-1)^k F_k^{q, \lambda}(u) \tau^k, \quad F_k^{q, \lambda}(u) = \sigma_k(q_1, \dots, q_N) + \sum_{s=1}^{\infty} F_{k,s}^{q, \lambda} u^{-s},$$

and $F_{k,s}^{q, \lambda} \in \mathcal{O}_\lambda$, $k = 1, \dots, N$, $s > 0$. In particular, we have

$$(5.6) \quad F_N^{q, \lambda}(u) = q_1 \dots q_N \frac{(u+1)^n + \sum_{s=1}^n (-1)^s \Sigma_s (u+1)^{n-s}}{u^n + \sum_{s=1}^n (-1)^s \Sigma_s u^{n-s}},$$

cf. (4.2).

Lemma 5.1. *The functions $F_{k,s}^{q, \lambda} \in \mathcal{O}_\lambda^q$, $k = 1, \dots, N$, $s > 0$, generate the algebra \mathcal{O}_λ^q .*

Proof. The coefficient of $u^{\lambda_i - j - 1}$ of the series $q_i^u \mathcal{D}^{\mathcal{O}_\lambda^q} f_i(u)$ has the form

$$-j p_{i,j} \prod_{\substack{k=1 \\ k \neq i}}^N (1 - q_k/q_i) + \sum_{l=1}^N \sum_{r=0}^{j+1} \sum_{s=0}^{j-1} c_{ijlrs} F_{l,r}^{q, \lambda} p_{i,s},$$

where c_{ijlrs} are some numbers, $F_{l,0}^{q, \lambda} = \sigma_l(q_1, \dots, q_N)$ and $p_{i,0} = 1$. Since $\mathcal{D}^{\mathcal{O}_\lambda^q} f_i(u) = 0$, we can express recursively the elements $p_{i,j}$ via the elements $F_{l,r}^{q, \lambda}$ starting with $j = 1$ and then increasing the second index j . \square

5.3. Discrete Wronski map π_λ^q . Consider \mathbb{C}^n with coordinates $\sigma_1, \dots, \sigma_n$. Introduce the discrete Wronski map $\pi_\lambda^q : \Omega_\lambda^q \rightarrow \mathbb{C}^n$ as follows. Let X be a point of Ω_λ^q . Define

$$(5.7) \quad \text{Wr}_X(u) = \text{Wr}(f_1(u-1, X), \dots, f_N(u-1, X)),$$

where $f_1(u, X), \dots, f_N(u, X)$ are given by (5.2). Let

$$\text{Wr}_X(u) = \prod_{i=1}^N q_i^{u-1} \prod_{1 \leq i < j \leq N} (q_j^{-1} - q_i^{-1}) \left(u^n + \sum_{s=1}^n (-1)^s a_s u^{n-s} \right).$$

We set $\pi_\lambda^q : X \mapsto (a_1, \dots, a_n)$.

The discrete Wronski map is a finite algebraic map, see [MTV5, Proposition 3.1]. It defines an injective algebra homomorphism

$$(\pi_{\lambda}^q)^* : \mathbb{C}[\sigma_1, \dots, \sigma_n] \rightarrow \mathcal{O}_{\lambda}^q, \quad \sigma_s \mapsto \Sigma_s,$$

which gives a $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module structure on \mathcal{O}_{λ}^q .

For $\mathbf{a} \in \mathbb{C}^n$, let $I_{\lambda, \mathbf{a}}^{\mathcal{O}}$ be the ideal in \mathcal{O}_{λ}^q generated by the elements $\Sigma_s - a_s$, $s = 1, \dots, n$, where $\Sigma_1, \dots, \Sigma_n$ are defined by (5.3). The quotient algebra

$$(5.8) \quad \mathcal{O}_{\lambda, \mathbf{a}}^q = \mathcal{O}_{\lambda}^q / I_{\lambda, \mathbf{a}}^{\mathcal{O}}$$

is the scheme-theoretic fiber of the discrete Wronski map π_{λ}^q .

5.4. First main result. Let \mathcal{A} be a commutative algebra. The algebra \mathcal{A} considered as a module over itself is called the regular representation of \mathcal{A} .

Theorem 5.2. *Assume that $\mathbf{q} \in (\mathbb{C}^{\times})^N$ has distinct coordinates. Denote $v_{\lambda} = \sum_{I \in \mathcal{I}_{\lambda}} v_I$. Then*

- (i) *The map $\mu_{\lambda}^q : F_{k,s}^{q, \lambda} \mapsto B_{k,s}^{q, \lambda}$, $k = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $\mu_{\lambda}^q : \mathcal{O}_{\lambda}^q \rightarrow \mathcal{B}^q((\mathcal{V}^S)_{\lambda})$ of filtered algebras. The isomorphism μ_{λ}^q becomes an isomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module \mathcal{O}_{λ}^q and the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^q((\mathcal{V}^S)_{\lambda})$ if we identify the algebras $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and $\mathbb{C}[z_1, \dots, z_n]^S$ by the map $\sigma_s \mapsto \sigma_s(\mathbf{z})$, $s = 1, \dots, n$.*
- (ii) *The map $\nu_{\lambda}^q : \mathcal{O}_{\lambda}^q \rightarrow (\mathcal{V}^S)_{\lambda}$, $f \mapsto \mu_{\lambda}^q(f) v_{\lambda}$, is an isomorphism of filtered vector spaces identifying the $\mathcal{B}^q((\mathcal{V}^S)_{\lambda})$ -module $(\mathcal{V}^S)_{\lambda}$ and the regular representation of \mathcal{O}_{λ}^q .*

The theorem is proved in Section 9.1.

Remark. Theorem 5.2 was announced in [GRTV, Theorem 6.3]. To indicate the correspondence of notation, we point out that formula (6.1) in [GRTV] is a counterpart of formula (5.3) in this paper with functions $g_i(u)$ in [GRTV] being equal to $q_i h^{\lambda_i} f_i(-1 + u/h)$ here. Also, formulae (6.3), (6.4) in [GRTV] correspond to formulae (5.4), (5.5) in this paper, and the algebra \mathcal{H}_{λ}^q in [GRTV] is a counterpart of the algebra \mathcal{O}_{λ}^q here.

Assume that the complex numbers b_1, \dots, b_n are such that $b_i \neq b_j + 1$ for $i > j$. Consider the tensor product $V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n)$ of evaluation $Y(\mathfrak{gl}_N)$ -modules and its weight subspace $V(\mathbf{b})_{\lambda}$. Introduce the numbers $\mathbf{a} = (a_1, \dots, a_n)$ by the formula $a_s = \sigma_s(b_1, \dots, b_n)$, cf. (3.4).

Corollary 5.3. *Assume that $\mathbf{q} \in (\mathbb{R}^{\times})^N$ has distinct coordinates. Let b_1, \dots, b_n be real and such that $|b_i - b_j| > 1$ for all $i \neq j$. Then the algebra $\mathcal{B}^q(V(\mathbf{b})_{\lambda})$ has simple spectrum.*

Proof. Under the assumption made, the algebra $\mathcal{B}^q(V(\mathbf{b})_{\lambda})$ has no nilpotent elements, see [MTV5, Lemma 3.7 and Lemma 3.10]. Hence the algebra $\mathcal{O}_{\lambda, \mathbf{a}}^q$, and thus the algebra $\mathcal{B}^q(V(\mathbf{b})_{\lambda})$, is the direct sum of one-dimensional algebras, so the spectrum of $\mathcal{B}^q(V(\mathbf{b})_{\lambda})$ is simple, see Proposition 3.5. \square

Other sufficient conditions for simplicity of the spectrum of $\mathcal{B}^q(V(\mathbf{b})_{\lambda})$ see in [MTV5, Theorem 2.1, part (2)] and in [MTV7, Theorem 1.1].

Corollary 5.4. *Assume that $\mathbf{q} \in (\mathbb{C}^\times)^N$ has distinct coordinates. Then the isomorphisms $\mu_\lambda^{\mathbf{q}}$ and $\nu_\lambda^{\mathbf{q}}$ induce an isomorphism of the $\mathcal{B}^{\mathbf{q}}(V(\mathbf{b})_\lambda)$ -module $V(\mathbf{b})_\lambda$ and the regular representation of the algebra $\mathcal{O}_{\lambda, \mathbf{a}}^{\mathbf{q}}$. \square*

The corollary implies that $\mathcal{B}^{\mathbf{q}}(V(\mathbf{b})_\lambda) \subset \text{End}(V(\mathbf{b})_\lambda)$ is a maximal commutative subalgebra and $\mathcal{B}^{\mathbf{q}}(V(\mathbf{b})_\lambda)$ is a Frobenius algebra, see for example [MTV4, Lemma 3.9].

6. SPACES OF POLYNOMIALS

6.1. Spaces of polynomials. Let $\lambda \in \mathbb{Z}_{\geq 0}^N$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, $|\lambda| = n$. In other words, let λ be a partition of n with at most N parts. Introduce the set $P = \{d_1 > d_2 > \dots > d_N\}$, where $d_i = \lambda_i + N - i$. Let Ω_λ be the affine n -dimensional space with coordinates $f_{i,j}$, $i = 1, \dots, N$, $j = 1, \dots, d_i$, $d_i - j \notin P$.

Introduce polynomials

$$(6.1) \quad f_i(u) = u^{d_i} + \sum_{\substack{j=1 \\ d_i-j \notin P}}^{d_i} f_{i,j} u^{d_i-j}, \quad i = 1, \dots, N.$$

We identify points $X \in \Omega_\lambda$ with N -dimensional complex vector spaces generated by polynomials

$$(6.2) \quad f_i(u, X) = u^{d_i} + \sum_{\substack{j=1 \\ d_i-j \notin P}}^{d_i} f_{i,j}(X) u^{d_i-j}, \quad i = 1, \dots, N.$$

Denote by \mathcal{O}_λ the algebra of regular functions on Ω_λ . It is the polynomial algebra in the variables $f_{i,j}$. The algebra \mathcal{O}_λ has the degree function such that $\deg f_{i,j} = j$ for all i, j . We consider the increasing filtration $F_0\mathcal{O}_\lambda \subset F_1\mathcal{O}_\lambda \subset \dots \subset \mathcal{O}_\lambda$, where $F_s\mathcal{O}_\lambda$ consists of elements of degree $\leq s$. The graded character of \mathcal{O}_λ is

$$\text{ch}_{\mathcal{O}_\lambda}(t) = \frac{\prod_{1 \leq i < j \leq N} (1 - t^{\lambda_i - \lambda_j + j - i})}{\prod_{i=1}^N (t)^{\lambda_i + N - i}}.$$

see [MTV4, Lemma 3.1].

6.2. Another realization of \mathcal{O}_λ . Let $f_i(u)$, $i = 1, \dots, N$, be the generating functions given by (6.1). We have

$$(6.3) \quad \text{Wr}(f_1(u-1), \dots, f_N(u-1)) = \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i + i - j) \left(u^n + \sum_{s=1}^n (-1)^s \Sigma_s u^{n-s} \right),$$

where $\Sigma_1, \dots, \Sigma_n$ are elements of \mathcal{O}_λ . Define the difference operator $\mathcal{D}^{\mathcal{O}_\lambda}$ by

$$(6.4) \quad \mathcal{D}^{\mathcal{O}_\lambda}(u, \tau) = \frac{1}{\text{Wr}(f_1(u-1), \dots, f_N(u-1))} \text{rdet} \begin{pmatrix} f_1(u) & f_1(u-1) & \dots & f_1(u-N) \\ f_2(u) & f_2(u-1) & \dots & f_2(u-N) \\ \dots & \dots & \dots & \dots \\ 1 & \tau & \dots & \tau^N \end{pmatrix}.$$

$$(6.5) \quad \mathcal{D}^{\mathcal{O}_\lambda}(u, \tau) \tau^{-N} = \sum_{k=0}^N (-1)^k G_k(u) (\tau^{-1} - 1)^{N-k}, \quad G_k(u) = \sum_{s=0}^{\infty} G_{k,s} u^{-s},$$

where $G_{k,s} \in \mathcal{O}_\lambda$.

Lemma 6.1. *The following statements hold.*

- (i) $G_0(u) = 1$.
- (ii) $G_{k,s} = 0$ for all $k = 1, \dots, N$ and $s < k$.
- (iii) $G_{1,1}, \dots, G_{N,N}$ are complex numbers, and for a variable x , we have For all x we have

$$\sum_{k=0}^N (-1)^k G_{k,k} \prod_{j=0}^{N-k-1} (x - j) = \prod_{s=1}^N (x - \lambda_s - N + s).$$

□

Lemma 6.2. *The functions $G_{k,s} \in \mathcal{O}_\lambda$, $k = 1, \dots, N$, $s = k+1, k+2, \dots$, generate the algebra \mathcal{O}_λ .*

Proof. The coefficient of u^{d_i-N-j} of the series $\mathcal{D}^{\mathcal{O}_\lambda} f_i(u)$ has the form $\chi(d_i - j) f_{i,j} + \dots$, where $\chi(x) = \prod_{s=1}^N (x - \lambda_s - N + s)$ and the dots denote the terms which contain the elements $G_{k,l}$ and $f_{i,s}$ with $s < j$ only. Since $\mathcal{D}^{\mathcal{O}_\lambda} f_i(u) = 0$ and $\chi(d_i - j) \neq 0$, we can express recursively the elements $f_{i,j}$ via the elements $G_{k,l}$ starting with $j = 1$ and then increasing the second index j . □

6.3. Discrete Wronski map π_λ . Consider \mathbb{C}^n with coordinates $\sigma_1, \dots, \sigma_n$. Introduce the discrete Wronski map $\pi_\lambda : \Omega_\lambda \rightarrow \mathbb{C}^n$ as follows. Let X be a point of Ω_λ . Define

$$(6.6) \quad \text{Wr}_X(u) = \text{Wr}(f_1(u-1, X), \dots, f_N(u-1, X)),$$

where $f_1(u, X), \dots, f_N(u, X)$ are given by (6.2). Let

$$\text{Wr}_X(u) = \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i + i - j) \left(u^n + \sum_{s=1}^n (-1)^s a_s u^{n-s} \right).$$

We set $\pi_\lambda : X \mapsto (a_1, \dots, a_n)$.

The discrete Wronski map is a finite algebraic map, see [MTV5, Proposition 3.1]. It defines an injective algebra homomorphism

$$(\pi_\lambda)^* : \mathbb{C}[\sigma_1, \dots, \sigma_n] \rightarrow \mathcal{O}_\lambda, \quad \sigma_s \mapsto \Sigma_s,$$

which gives a $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module structure on \mathcal{O}_λ^q .

For $\mathbf{a} \in \mathbb{C}^n$, let $I_{\lambda, \mathbf{a}}^\mathcal{O}$ be the ideal in \mathcal{O}_λ generated by the elements $\Sigma_s - a_s$, $s = 1, \dots, n$, where $\Sigma_1, \dots, \Sigma_n$ are defined by (6.3). The quotient algebra

$$(6.7) \quad \mathcal{O}_{\lambda, \mathbf{a}} = \mathcal{O}_\lambda / I_{\lambda, \mathbf{a}}^\mathcal{O}$$

is the scheme-theoretic fiber of the discrete Wronski map π_λ .

6.4. Second main result. By formula (2.3), the space $F_k(\mathcal{V}^S)_\lambda^{sing}$ is one-dimensional if $k = \sum_{i=1}^N (i-1)\lambda_i$. Fix a nonzero element $v_\lambda^{sing} \in (\mathcal{V}^S)_\lambda^{sing}$ in that subspace.

Theorem 6.3. *Let λ be a partition on n with at most N parts. Then*

- (i) *The map $\mu_\lambda : G_{k,s} \mapsto C_{k,s}^\lambda$, $k = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $\mu_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{sing})$ of filtered algebras. The isomorphism μ_λ becomes an isomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module \mathcal{O}_λ and the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{sing})$ if we identify the algebras $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and $\mathbb{C}[z_1, \dots, z_n]^S$ by the map $\sigma_s \mapsto \sigma_s(\mathbf{z})$, $s = 1, \dots, n$.*
- (ii) *The map $\nu_\lambda : \mathcal{O}_\lambda \rightarrow (\mathcal{V}^S)_\lambda^{sing}$, $f \mapsto \mu_\lambda(f) v_\lambda^{sing}$, is an isomorphism of filtered vector spaces increasing the index of filtration by k_{min} . The isomorphism ν_λ identifies the $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{sing})$ -module $(\mathcal{V}^S)_\lambda^{sing}$ and the regular representation of the algebra \mathcal{O}_λ .*

The theorem is proved in Section 9.2.

Assume that the complex numbers b_1, \dots, b_n are such that $b_i \neq b_j + 1$ for $i > j$. Consider the tensor product $V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n)$ of evaluation $Y(\mathfrak{gl}_N)$ -modules and its singular weight subspace $V(\mathbf{b})_\lambda^{sing}$. Introduce the numbers $\mathbf{a} = (a_1, \dots, a_n)$ by the formula $a_s = \sigma_s(b_1, \dots, b_n)$, cf. (3.4).

Corollary 6.4. *The isomorphisms μ_λ and ν_λ induce an isomorphism of the $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing})$ -module $V(\mathbf{b})_\lambda^{sing}$ and the regular representation of the algebra $\mathcal{O}_{\lambda, \mathbf{a}}$. In particular, $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing}) \subset \text{End}(V(\mathbf{b})_\lambda^{sing})$ is a maximal commutative subalgebra and $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing})$ is a Frobenius algebra, see for example [MTV4, Lemma 3.9]. \square*

Remark. Corollary 6.4 is used in [MTV6] to prove Theorem 7.3 therein, similarly to the proof of Theorems 4.1, 4.3 in [MTV6]. The algebra $\mathcal{B}_{n,N,\lambda}^Y(b_1, \dots, b_n)$ in [MTV6] coincides with the algebra $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing})$ in this paper.

Corollary 6.5. *Let b_1, \dots, b_n be real and such that $|b_i - b_j| > 1$ for all $i \neq j$. Then the algebra $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing})$ has simple spectrum. \square*

The proof is similar to that of Corollary 5.3.

Other sufficient conditions for simplicity of the spectrum of $\mathcal{B}^{q=1}(V(\mathbf{b})_\lambda^{sing})$ see in [MTV5, Theorem 2.1, part (2)] and in [MTV7, Theorem 1.1].

7. BETHE ANSATZ FOR \mathbf{q} WITH DISTINCT COORDINATES

To prove Theorems 5.2 and 6.3 we need some facts about the Bethe ansatz. We consider the tensor product of evaluation $Y(\mathfrak{gl}_N)$ -modules $V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n)$ and the action of the Bethe algebra \mathcal{B}^q on the weight subspace $V(\mathbf{b})_\lambda$.

7.1. Bethe ansatz equations associated with $V(\mathbf{b})_\lambda$. Recall $\lambda = (\lambda_1, \dots, \lambda_N)$, $|\lambda| = n$. Introduce $\mathbf{l} = (l_1, \dots, l_{N-1})$ with $l_j = \lambda_{j+1} + \dots + \lambda_N$. We have $n \geq l_1 \geq \dots \geq l_{N-1} \geq 0$.

Set $l_0 = n$, $l_N = 0$, and $l = l_1 + \dots + l_{N-1}$. We shall consider functions of l variables

$$\mathbf{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}, t_1^{(2)}, \dots, t_{l_2}^{(2)}, \dots, t_1^{(N-1)}, \dots, t_{l_{N-1}}^{(N-1)}).$$

The following system of l algebraic equations with respect to l variables \mathbf{t} is called the Bethe ansatz equations associated with $V(\mathbf{b})_\lambda$ and \mathbf{q} :

$$\begin{aligned} (7.1) \quad & q_1 \prod_{s=1}^n (t_j^{(1)} - b_s + 1) \prod_{\substack{j'=1 \\ j' \neq j}}^{l_1} (t_j^{(1)} - t_{j'}^{(1)} - 1) \prod_{j'=1}^{l_2} (t_j^{(1)} - t_{j'}^{(2)}) = \\ & = q_2 \prod_{s=1}^n (t_j^{(1)} - b_s) \prod_{\substack{j'=1 \\ j' \neq j}}^{l_1} (t_j^{(1)} - t_{j'}^{(1)} + 1) \prod_{j'=1}^{l_2} (t_j^{(1)} - t_{j'}^{(2)} - 1), \\ & q_a \prod_{j'=1}^{l_{a-1}} (t_j^{(a)} - t_{j'}^{(a-1)} + 1) \prod_{\substack{j'=1 \\ j' \neq j}}^{l_a} (t_j^{(a)} - t_{j'}^{(a)} - 1) \prod_{j'=1}^{l_{a+1}} (t_j^{(a)} - t_{j'}^{(a+1)}) = \\ & = q_{a+1} \prod_{j'=1}^{l_{a-1}} (t_j^{(a)} - t_{j'}^{(a-1)}) \prod_{\substack{j'=1 \\ j' \neq j}}^{l_a} (t_j^{(a)} - t_{j'}^{(a)} + 1) \prod_{j'=1}^{l_{a+1}} (t_j^{(a)} - t_{j'}^{(a+1)} - 1). \end{aligned}$$

Here the equations of the first group are labeled by $j = 1, \dots, l_1$, the equations of the second group are labeled by $a = 2, \dots, N-1$, $j = 1, \dots, l_a$,

A solution $\tilde{\mathbf{t}}$ of system (7.1) is called off-diagonal if $\tilde{t}_j^{(a)} \neq \tilde{t}_{j'}^{(a)}$ for any $a = 1, \dots, N-1$, $1 \leq j \leq j' \leq l_a$, and $\tilde{t}_j^{(a)} \neq \tilde{t}_{j'}^{(a+1)}$ for any $a = 1, \dots, N-2$, $j = 1, \dots, l_a$, $j' = 1, \dots, l_{a+1}$.

Remark. If $\lambda = (n, 0, \dots, 0)$, then $l_1 = \dots = l_{N-1} = 0$. In this case, there are no variables \mathbf{t} and it is convenient to think that the Bethe ansatz equations is the equation $1 = 1$.

7.2. Weight function and Bethe ansatz theorem. Denote by $\omega_\lambda(\mathbf{t}, \mathbf{b})$ the universal weight function associated with the weight subspace $V(\mathbf{b})_\lambda$. The universal weight function is defined by formula (6.2) in [MTV1], see explicit formula (7.7) below, cf. [TV1, MTV2, RTV, TV3]. For the moment, it is enough for us to know that this function is a $V(\mathbf{b})_\lambda$ -valued polynomial in \mathbf{t} , \mathbf{b} .

If $\tilde{\mathbf{t}}$ is an off-diagonal solution of the Bethe ansatz equations, then the vector $\omega_\lambda(\tilde{\mathbf{t}}, \mathbf{b}) \in V(\mathbf{b})_\lambda$ is called the Bethe vector associated with $\tilde{\mathbf{t}}$.

Theorem 7.1. *Let $\tilde{\mathbf{t}}$ be an off-diagonal solution of the Bethe ansatz equations (7.1). Assume that the Bethe vector $\omega_\lambda(\tilde{\mathbf{t}}, \mathbf{b})$ is nonzero. Then the Bethe vector is an eigenvector of all transfer-matrices $B_k^q(u)$, $k = 1, \dots, N$. \square*

The statement follows from Theorem 6.1 in [MTV1]. For $k = 1$, the result is established in [KR1].

The eigenvalues of the Bethe vector are as follows. Set

$$\begin{aligned}\chi_1(u, \mathbf{t}, \mathbf{b}) &= q_1 \prod_{s=1}^n \frac{u - b_s + 1}{u - b_s} \prod_{j=1}^{l_1} \frac{u - t_j^{(1)} - 1}{u - t_j^{(1)}}, \\ \chi_a(u, \mathbf{t}, \mathbf{b}) &= q_a \prod_{j=1}^{l_{a-1}} \frac{u - t_j^{(a-1)} + 1}{u - t_j^{(a-1)}} \prod_{j=1}^{l_a} \frac{u - t_j^{(a)} - 1}{u - t_j^{(a)}},\end{aligned}$$

for $a = 2, \dots, N$. Define the functions $c_k(u, \mathbf{t}, \mathbf{b})$ by the formula

$$(7.2) \quad (1 - \chi_1(u, \mathbf{t}, \mathbf{b}) \tau) \cdots (1 - \chi_N(u, \mathbf{t}, \mathbf{b}) \tau) = \sum_{k=0}^N (-1)^k c_k(u, \mathbf{t}, \mathbf{b}) \tau^k.$$

Then

$$(7.3) \quad B_k^q(u) \omega_{\lambda}(\tilde{\mathbf{t}}, \mathbf{b}) = c_k(u, \tilde{\mathbf{t}}, \mathbf{b}) \omega_{\lambda}(\tilde{\mathbf{t}}, \mathbf{b})$$

for $k = 1, \dots, N$, see Theorem 6.1 in [MTV1].

Remark. If $\lambda = (n, 0, \dots, 0)$, then $V(\mathbf{b})_{\lambda}$ is the one-dimensional space generated by the vector $v_1 \otimes \dots \otimes v_1$. It is convenient to assume that the universal weight function is given by the formula $\omega(\mathbf{b}) = v_1 \otimes \dots \otimes v_1$. This vector is an eigenvector of the Bethe algebra \mathcal{B}^q . The eigenvalues of this eigenvector are defined by formula (7.2), in which the difference operator takes the form

$$(7.4) \quad \left(1 - q_1 \left(\prod_{s=1}^n \frac{u - b_s + 1}{u - b_s}\right) \tau\right) \prod_{i=2}^N (1 - q_i \tau).$$

7.3. Difference operator associated with an off-diagonal solution. For $\tilde{\mathbf{t}} \in \mathbb{C}^l$, we introduce the associated fundamental difference operator

$$(7.5) \quad \mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau) = \sum_{k=0}^N (-1)^k c_k(u, \tilde{\mathbf{t}}, \mathbf{b}) \tau^k,$$

see [MTV2]. Here the functions $c_k(u, \tilde{\mathbf{t}}, \mathbf{b})$ are given by (7.2).

Theorem 7.2. Assume that \mathbf{q} has distinct coordinates. Let $\tilde{\mathbf{t}}$ be an off-diagonal solution of the Bethe ansatz equations. Then there exist polynomials $p_k(u)$, $k = 1, \dots, N$, such that $\deg p_k(u) = \lambda_k$, $\mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau) q_k^u p_k(u) = 0$, and

$$(7.6) \quad \text{Wr}(q_1^{u-1} p_1(u-1), \dots, q_N^{u-1} p_N(u-1)) = \prod_{i=1}^N q_i^{u-1} \prod_{1 \leq i < j \leq N} (q_j^{-1} - q_i^{-1}) \prod_{s=1}^n (u - b_s).$$

□

This is Proposition 7.6 in [MV3], which is a generalization of Lemma 4.8 in [MV1].

Remark. If $\lambda = (n, 0, \dots, 0)$, then $V(\mathbf{b})_{\lambda}$ is spanned by $v_1 \otimes \dots \otimes v_1$. The corresponding fundamental difference operator $\mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau)(u, \tau)$ is given by (7.4). The polynomials $p_2(u), \dots, p_N(u)$ of Theorem 7.2 are just constants and the polynomial $p_1(u)$ is uniquely determined (up to proportionality) by the condition $\mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau)(u, \tau) q_1^u p_1(u) = 0$.

7.4. Completeness of the Bethe ansatz.

Theorem 7.3. *Assume that \mathbf{q} has distinct coordinates. Then for any $\boldsymbol{\lambda}$ and generic b_1, \dots, b_n , there exists a collection of off-diagonal solutions of the Bethe ansatz equations such that the corresponding Bethe vectors form a basis of $V(\mathbf{b})_{\boldsymbol{\lambda}}$.*

Theorem 7.3 is proved in Sections 7.6 and 7.7.

7.5. Weight functions W_I . For a function $f(t_1, \dots, t_k)$ of some variables, denote

$$\text{Sym}_{t_1, \dots, t_k} f(t_1, \dots, t_k) = \sum_{\sigma \in S_k} f(t_{\sigma_1}, \dots, t_{\sigma_k}).$$

Recall $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ and $I = (I_1, \dots, I_N)$. Set $\bigcup_{c=a+1}^N I_c = \{i_1^{(a)} < \dots < i_{l_a}^{(a)}\}$. Introduce $t^{(0)} = (t_1^{(0)}, \dots, t_n^{(0)}) = (b_1, \dots, b_n)$.

For $I \in \mathcal{I}_{\boldsymbol{\lambda}}$, we define the weight functions $W_I(\mathbf{t}; \mathbf{b})$, cf. [TV1, TV3]:

$$(7.7) \quad W_I(\mathbf{t}; \mathbf{b}) = \text{Sym}_{t_1^{(1)}, \dots, t_{l_1}^{(1)}} \dots \text{Sym}_{t_1^{(N-1)}, \dots, t_{l_{N-1}}^{(N-1)}} U_I(\mathbf{t}; \mathbf{b}),$$

$$U_I(\mathbf{t}; \mathbf{b}) = \prod_{a=1}^{N-1} \prod_{j=1}^{l_a} \left(\prod_{\substack{j'=1 \\ i_{j'}^{(a-1)} < i_j^{(a)}}}^{l_{a-1}} (t_j^{(a)} - t_{j'}^{(a-1)} + 1) \prod_{\substack{j'=1 \\ i_{j'}^{(a-1)} > i_j^{(a)}}}^{l_{a-1}} (t_j^{(a)} - t_{j'}^{(a-1)}) \prod_{j'=j+1}^{l_a} \frac{t_j^{(a)} - t_{j'}^{(a)} + 1}{t_j^{(a)} - t_{j'}^{(a)}} \right).$$

The universal $V(\mathbf{b})_{\boldsymbol{\lambda}}$ -valued weight function is the function

$$(7.8) \quad \omega_{\boldsymbol{\lambda}}(\mathbf{t}, \mathbf{b}) = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} W_I(\mathbf{t}, \mathbf{b}) v_I \in V(\mathbf{b})_{\boldsymbol{\lambda}}.$$

7.6. Proof of Theorem 7.3 for $n = 1$. If $n = 1$, then $\boldsymbol{\lambda} = (0, \dots, 0, 1_{k+1}, 0, \dots, 0)$, where 1 is at the $k + 1$ -st position. If $k = 0$, Theorem 7.3 holds due to remarks in Sections 7.1 and 7.2.

Assume $k > 0$. Then $\mathbf{t} = (t_1^{(1)}, t_1^{(2)}, \dots, t_1^{(k)})$ and $\omega_{\boldsymbol{\lambda}}(\mathbf{t}, \mathbf{b}) = v_{k+1}$. The Bethe ansatz equations are

$$(7.9) \quad \begin{aligned} q_1 (t_1^{(1)} - b_1 + 1) (t_1^{(1)} - t_1^{(2)}) &= q_2 (t_1^{(1)} - b_1) (t_1^{(1)} - t_1^{(2)} - 1), \\ q_a (t_1^{(a)} - t_1^{(a-1)} + 1) (t_1^{(a)} - t_1^{(a+1)}) &= q_{a+1} (t_1^{(a)} - t_1^{(a-1)}) (t_1^{(a)} - t_1^{(a+1)} - 1), \\ q_k (t_1^{(k)} - t_1^{(k-1)} + 1) &= q_{k+1} (t_1^{(k)} - t_1^{(k-1)}). \end{aligned}$$

Here the equations of the second group are labeled by $a = 2, \dots, k - 1$. The Bethe ansatz equations have the unique solution

$$(7.10) \quad t_1^{(i)} = b_1 + \sum_{j=1}^i \frac{q_j}{q_{k+1} - q_j}, \quad i = 1, \dots, k.$$

This solutions is off-diagonal. Theorem 7.3 for $n = 1$ is proved.

7.7. Proof of Theorem 7.3 for $n > 1$. Assume that b_1, \dots, b_n depend on a parameter $y \in \mathbb{C}$, so that $b_s(y) = sy$. The next lemma implies Theorem 7.3.

Lemma 7.4. *For $I \in \mathcal{I}_\lambda$, there exists an off-diagonal solution $\tilde{\mathbf{t}}(y)$ of the Bethe ansatz equations (7.1) such that the line generated by the Bethe vector $\omega(\tilde{\mathbf{t}}(y), \mathbf{b}(y))$ tends to the line generated by the vector v_I as y tends to infinity.*

Proof. To simplify the notations we consider an example. The general case is similar. Assume that $n = 2$ and $v_I = v_3 \otimes v_2$. Then $\mathbf{t} = (t_1^{(1)}, t_2^{(1)}, t_1^{(2)})$. We look for a solution of the Bethe ansatz equations in the form

$$(7.11) \quad t_1^{(i)} = b_1(y) + v_1^{(i)}(y), \quad i = 1, 2, \quad \text{and} \quad t_2^{(1)} = b_2(y) + v_2^{(1)}(y).$$

Then the Bethe ansatz equations take the form

$$(7.12) \quad \frac{v_2^{(1)} + 1}{v_2^{(1)}} = \frac{q_2}{q_1} + O(y^{-1}), \quad \frac{v_1^{(1)} + 1}{v_1^{(1)}} \cdot \frac{v_1^{(1)} - v_1^{(2)}}{v_1^{(1)} - v_1^{(2)} - 1} = \frac{q_2}{q_1} + O(y^{-1}),$$

$$\frac{v_1^{(2)} - v_1^{(1)} + 1}{v_1^{(2)} - v_1^{(1)}} = \frac{q_3}{q_2} + O(y^{-1}).$$

As $y \rightarrow \infty$, this system of three equations splits into an equation assigned to b_1 and a system of two equations assigned to b_2 according to our choice (7.11). Each of the limiting systems is the system (7.9) of the Bethe ansatz equations for $n = 1$ considered in Section 7.6. System (7.9) has a unique solution (7.10). By deforming that solution, we obtain a solution $\tilde{\mathbf{v}}(y)$ of system (7.1) whose limit as $y \rightarrow \infty$ equals

$$\left(\frac{q_1}{q_3 - q_1}, \frac{q_1}{q_2 - q_1}, \frac{q_1}{q_3 - q_1} + \frac{q_2}{q_3 - q_2} \right),$$

see Section 7.6. Clearly, $\tilde{\mathbf{v}}(y)$ corresponds to an off-diagonal solution $\tilde{\mathbf{t}}(y)$ of the Bethe ansatz equations as $y \rightarrow \infty$. It is easy to see that the limit of the line generated by the Bethe vector $\omega(\tilde{\mathbf{t}}(y), \mathbf{b}(y))$ as $y \rightarrow \infty$ is the line generated by the vector $v_3 \otimes v_2$, see formula (7.7). \square

8. BETHE ANSATZ FOR $\mathbf{q} = \mathbf{1}$

We consider the action of the Bethe algebra $\mathcal{B}^{\mathbf{q}=\mathbf{1}}$ on the subspace $V(\mathbf{b})_\lambda^{\text{sing}}$ of singular vectors. We use the notation of Section 7.

For $\mathbf{q} = \mathbf{1}$, the Bethe ansatz equations are given by (7.1) with $q_1 = \dots = q_N = 1$. Let $\omega_\lambda(\mathbf{t}, \mathbf{b})$ be the universal weight function associated with the weight subspace $V(\mathbf{b})_\lambda$, see (7.8).

Theorem 8.1. *Let $\tilde{\mathbf{t}}$ be an off-diagonal solution of the Bethe ansatz equations for $\mathbf{q} = \mathbf{1}$. Then $\omega_\lambda(\tilde{\mathbf{t}}, \mathbf{b})$ lies in $V(\mathbf{b})_\lambda^{\text{sing}}$.* \square

The statement is Lemma 5.3 in [TV2] or Proposition 6.2 in [MTV1]. For $N = 2$, the result is established in [FT], and for $N = 3$ in [KR2].

Theorem 8.2 ([MV1, Lemma 4.8]). *Let $\tilde{\mathbf{t}}$ be an off-diagonal solution of the Bethe ansatz equations for $\mathbf{q} = \mathbf{1}$. Consider the associated fundamental difference operator $\mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau)$,*

see (7.5). There exist polynomials $f_k(u) \in \mathbb{C}[u]$, $k = 1, \dots, N$, of the form described in (6.1), such that $\deg f_k(u) = \lambda_k + N - k$, $\mathcal{D}_{\tilde{\mathbf{t}}}(u, \tau)(u, \tau)f_k(u) = 0$, and

$$(8.1) \quad \text{Wr}(f_1(u-1), \dots, f_N(u-1)) = \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i + i - j) \prod_{s=1}^n (u - b_i). \quad \square$$

Theorem 8.3. *For generic b_1, \dots, b_n there exists a collection of off-diagonal solutions of the Bethe ansatz equations for $\mathbf{q} = \mathbf{1}$ such that the corresponding Bethe vectors form a basis of $V(\mathbf{b})_{\lambda}^{\text{sing}}$.*

Proof. Assume that b_1, \dots, b_n depend on a parameter $y \in \mathbb{C}$, so that $b_s = yd_s$ for given d_1, \dots, d_n , and y tends to infinity. We look for a solution of equations (7.1) for $\mathbf{q} = \mathbf{1}$ in the form $t_j^{(i)} = yv_j^{(i)}$. Then the equations are

$$(8.2) \quad \sum_{s=1}^n \frac{1}{v_j^{(1)} - d_s} - \sum_{\substack{j'=1 \\ j' \neq j}}^{l_1} \frac{2}{v_j^{(1)} - v_{j'}^{(1)}} + \sum_{j'=1}^{l_2} \frac{1}{v_j^{(1)} - v_{j'}^{(2)}} = 0 + O(y^{-1}),$$

$$\sum_{j'=1}^{l_{a-1}} \frac{1}{v_j^{(a)} - v_{j'}^{(a-1)}} - \sum_{\substack{j'=1 \\ j' \neq j}}^{l_a} \frac{2}{v_j^{(a)} - v_{j'}^{(a)}} + \sum_{j'=1}^{l_{a+1}} \frac{1}{v_j^{(a)} - v_{j'}^{(a+1)}} = 0 + O(y^{-1}).$$

In the limit $y \rightarrow \infty$, equations (8.2) tend to the Bethe ansatz equations of the Gaudin model associated with $V(\mathbf{d})_{\lambda}^{\text{sing}}$, considered in [MV2]. By [MV2, Theorem 6.1], for generic $\mathbf{d} = (d_1, \dots, d_n)$ there exists a collection of off-diagonal multiplicity-free solutions $\tilde{\mathbf{v}}$ of system (8.2) at $y = \infty$ such that the corresponding Bethe vectors $\omega_{\lambda}^{\text{Gaudin}}(\tilde{\mathbf{v}}, \mathbf{d})$ of the Gaudin model, see formula (4) in [MV2], form a basis of $V_{\lambda}^{\text{sing}}$. By deforming these solutions, we get a collection of solutions $\tilde{\mathbf{v}}(y)$ of system (8.2), and hence a collection of off-diagonal solutions $\tilde{\mathbf{t}}(y) = y\tilde{\mathbf{v}}(y)$ of system (7.1) for $\mathbf{q} = \mathbf{1}$. It easily follows from formula (7.7) and formula (4) in [MV2] that the lines generated by the Bethe vectors $\omega_{\lambda}(\tilde{\mathbf{t}}(y), \mathbf{b}(y))$ tend respectively to the lines generated by the Bethe vectors $\omega_{\lambda}^{\text{Gaudin}}(\tilde{\mathbf{v}}, \mathbf{d})$. Hence the Bethe vectors $\omega_{\lambda}(\tilde{\mathbf{t}}(y), \mathbf{b}(y))$ form a basis of $V(\mathbf{b})_{\lambda}^{\text{sing}}$ as $y \rightarrow \infty$. This proves Theorem 8.3. \square

9. PROOFS OF THEOREMS 5.2 AND 6.3

9.1. Proof of Theorem 5.2. Let a polynomial $R(F_{k,s}^{\mathbf{q}, \lambda})$ in generators $F_{k,s}^{\mathbf{q}, \lambda}$ equal zero in $\mathcal{O}_{\lambda}^{\mathbf{q}}$. Consider $R(B_{k,s}^{\mathbf{q}, \lambda})$ as a polynomial in z_1, \dots, z_n with values in $\text{End}((V^{\otimes n})_{\lambda})$. By Theorems 7.2 and 7.3, this polynomial equals zero for generic values of z_1, \dots, z_n . Hence, $R(F_{k,s}^{\mathbf{q}, \lambda})$ equals zero identically and the map $\mu_{\lambda}^{\mathbf{q}}: \mathcal{O}_{\lambda}^{\mathbf{q}} \rightarrow \mathcal{B}^{\mathbf{q}}((\mathcal{V}^S)_{\lambda})$ is well-defined.

Let a polynomial $R(F_{k,s}^{\mathbf{q}, \lambda})$ be a nonzero element of $\mathcal{O}_{\lambda}^{\mathbf{q}}$. By Theorems 7.2 and 7.3, it means that $R(B_{k,s}^{\mathbf{q}, \lambda})$ is nonzero in $\mathcal{B}^{\mathbf{q}}((\mathcal{V}^S)_{\lambda})$. This shows that $\mu_{\lambda}^{\mathbf{q}}$ is injective. Since the elements $B_{k,s}^{\mathbf{q}, \lambda}$ generate the algebra $\mathcal{B}^{\mathbf{q}}((\mathcal{V}^S)_{\lambda})$, the map $\mu_{\lambda}^{\mathbf{q}}$ is surjective. By comparing formulae (4.2) and (5.6), we conclude that $\mu_{\lambda}^{\mathbf{q}}$ is a homomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module $\mathcal{O}_{\lambda}^{\mathbf{q}}$ to the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^{\mathbf{q}}((\mathcal{V}^S)_{\lambda})$. Since $\deg F_{k,s}^{\mathbf{q}, \lambda} = \deg B_{k,s}^{\mathbf{q}, \lambda}$, the homomorphism $\mu_{\lambda}^{\mathbf{q}}$ is filtered. These remarks prove part (i) of Theorem 5.2.

Consider the map $\nu_\lambda^q : \mathcal{O}_\lambda^q \rightarrow (\mathcal{V}^S)_\lambda$, $f \mapsto \mu_\lambda^q(f) v_\lambda$. The kernel of ν_λ^q is an ideal in \mathcal{O}_λ^q which has zero intersection with $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and, therefore, is the zero ideal. Since $\text{ch}_{\mathcal{O}_\lambda^q}(t) = \text{ch}_{(\mathcal{V}^S)_\lambda}(t)$, we conclude that ν_λ^q is a linear isomorphism. This gives part (ii) of Theorem 5.2. \square

9.2. Proof of Theorem 6.3. The proof of Theorem 6.3 is similar to the proof of Theorem 5.2.

Namely, let a polynomial $R(G_{k,s})$ in generators $G_{k,s}$ equal zero in \mathcal{O}_λ . Consider $R(C_{k,s})$ as a polynomial in z_1, \dots, z_n with values in $\text{End}((V^{\otimes n})_\lambda^{\text{sing}})$. By Theorems 8.2 and 8.3, this polynomial equals zero for generic values of z_1, \dots, z_n . Hence, $R(C_{k,s})$ equals zero identically and the map $\mu_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{\text{sing}})$ is well-defined.

Let a polynomial $R(G_{k,s})$ be a nonzero element of \mathcal{O}_λ . By Theorems 8.2 and 8.3, it means that $R(C_{k,s})$ is nonzero in $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{\text{sing}})$. This shows that μ_λ is injective. Since the elements $C_{k,s}$ generate the algebra $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{\text{sing}})$, the map μ_λ is surjective. By comparing formulae (4.2) and (5.6), we conclude that μ_λ is a homomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module \mathcal{O}_λ to the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^{q=1}((\mathcal{V}^S)_\lambda^{\text{sing}})$. Since $\deg G_{k,s} = \deg C_{k,s}$, the homomorphism μ_λ is filtered. These remarks prove part (i) of Theorem 6.3.

Consider the map $\nu_\lambda : \mathcal{O}_\lambda \rightarrow (\mathcal{V}^S)_\lambda^{\text{sing}}$, $f \mapsto \mu_\lambda(f) v_\lambda$. The kernel of ν_λ is an ideal in \mathcal{O}_λ which has zero intersection with $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and, therefore, is the zero ideal. Since $t^{\sum_{i=1}^N (i-1)\lambda_i} \text{ch}_{\mathcal{O}_\lambda}(t) = \text{ch}_{(\mathcal{V}^S)_\lambda^{\text{sing}}}(t)$, we conclude that ν_λ is a linear isomorphism. This gives part (ii) of Theorem 6.3. \square

10. SPACE $\frac{1}{D}\mathcal{V}^A$

10.1. Definitions. Recall $\mathcal{V} = V \otimes \mathbb{C}[z_1, \dots, z_n]$ and the S_n -action on V -valued functions of z_1, \dots, z_n defined by formula (2.1). We denote by \mathcal{V}^A the subspace of the S_n -skew-invariants in \mathcal{V} . The space \mathcal{V}^A is a filtered space.

Let

$$D = \prod_{1 \leq i < j \leq n} (z_j - z_i + 1).$$

We denote by $\frac{1}{D}\mathcal{V}$ the space of V -valued functions of z_1, \dots, z_n of the form $\frac{1}{D}f$, $f \in \mathcal{V}$, and by $\frac{1}{D}\mathcal{V}^A$ the space of V -valued functions of z_1, \dots, z_n of the form $\frac{1}{D}f$, $f \in \mathcal{V}^A$.

Lemma 10.1 ([GRTV, Lemma 2.9]). *A V -valued function f of z_1, \dots, z_n is skew-invariant with respect to the S_n -action if and only if the function $\frac{1}{D}f$ is invariant with respect to the S_n -action.* \square

By this lemma, we can define the space $\frac{1}{D}\mathcal{V}^A$ as the space of V -valued S_n -invariant functions of z_1, \dots, z_n of the form $\frac{1}{D}f$, $f \in \mathcal{V}$.

Lemma 10.2. *The space $\frac{1}{D}\mathcal{V}^A$ is a free $\mathbb{C}[z_1, \dots, z_n]^S$ -module of rank N^n .*

Proof. The lemma follows from Lemma 2.10 in [GRTV]. \square

We define the degree of elements $\frac{1}{D}f \in \frac{1}{D}\mathcal{V}^A$ by the formula $\deg(\frac{1}{D}f) = \deg(f) - n(n-1)/2$. We consider the increasing filtration $\dots \subset F_{k-1}\frac{1}{D}\mathcal{V}^A \subset F_k\frac{1}{D}\mathcal{V}^A \subset \dots \subset \frac{1}{D}\mathcal{V}^A$ whose k -th subspace consists of elements of degree $\leq k$.

The space $\frac{1}{D}\mathcal{V}^A$ is a filtered \mathfrak{gl}_N -module. We consider the \mathfrak{gl}_N -weight decomposition

$$\frac{1}{D}\mathcal{V}^A = \bigoplus_{\substack{\lambda \in \mathbb{Z}_{\geq 0}^N \\ |\lambda| = n}} (\frac{1}{D}\mathcal{V}^A)_\lambda,$$

as well as the subspaces of singular vectors $(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing} \subset (\frac{1}{D}\mathcal{V}^A)_\lambda$. All of these are filtered free $\mathbb{C}[z_1, \dots, z_n]^S$ -modules.

Lemma 10.3. *For $\lambda \in \mathbb{Z}_{\geq 0}^N$, $|\lambda| = n$, we have*

$$(10.1) \quad \text{ch}_{(\frac{1}{D}\mathcal{V}^A)_\lambda}(t) = t^{-\sum_{1 \leq i < j \leq N} \lambda_i \lambda_j} \prod_{i=1}^N \frac{1}{(t)_{\lambda_i}}.$$

For a partition λ of n with at most N parts, we have

$$(10.2) \quad \text{ch}_{(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}}(t) = t^{-\sum_{1 \leq i < j \leq N} \lambda_i \lambda_j} \frac{\prod_{1 \leq i < j \leq N} (1 - t^{\lambda_i - \lambda_j + j - i})}{\prod_{i=1}^N (t)_{\lambda_i + N - i}}.$$

Proof. The graded components $F_k(\frac{1}{D}\mathcal{V}^A)_\lambda / F_{k-1}(\frac{1}{D}\mathcal{V}^A)_\lambda$ and $F_k(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing} / F_{k-1}(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}$ are respectively naturally isomorphic to the graded components considered in [RSTV] and denoted there by $((\frac{1}{D}\mathcal{V}^-)_\lambda)_k$ and $((\frac{1}{D}\mathcal{V}^-)_\lambda^{sing})_k$. Now formula (10.1) follows from Theorem [RSTV, Theorem 3.4] and [MTV3, Lemma 2.12]. Formula (10.2) follows from [RSTV, Formula 3.4]. \square

10.2. Space $\frac{1}{D}\mathcal{V}^A$ as a Yangian module. By Lemmas 3.2 and 3.3, $\frac{1}{D}\mathcal{V}^A$ is a filtered $Y(\mathfrak{gl}_N)$ -module.

For $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n$, we define $\mathbf{a} = (a_1, \dots, a_n)$ by the formula $a_s = \sigma_s(b_1, \dots, b_n)$, cf. (3.4).

Proposition 10.4. *Assume that the numbers b_1, \dots, b_n are such that $b_i \neq b_j + 1$ for all $i \neq j$. Then the $Y(\mathfrak{gl}_N)$ -module $\frac{1}{D}\mathcal{V}^A / I_{\mathbf{a}} \frac{1}{D}\mathcal{V}^A$ is isomorphic to $V(\mathbf{b}) = \mathbb{C}^N(b_1) \otimes \dots \otimes \mathbb{C}^N(b_n)$, the tensor product of evaluation $Y(\mathfrak{gl}_N)$ -modules.*

Proof. Consider the map $\varphi^A: \frac{1}{D}\mathcal{V}^A \rightarrow V(\mathbf{b})$ that sends every element of $\frac{1}{D}\mathcal{V}^A$ to its value at the point $\mathbf{z} = \mathbf{b}$. This map is a homomorphism of $Y(\mathfrak{gl}_N)$ -modules and factors through the canonical projection $\vartheta^A: \frac{1}{D}\mathcal{V}^A \rightarrow \frac{1}{D}\mathcal{V}^A / I_{\mathbf{a}} \frac{1}{D}\mathcal{V}^A$. Since ϑ^A is also a homomorphism of $Y(\mathfrak{gl}_N)$ -modules, this defines a homomorphism of $Y(\mathfrak{gl}_N)$ -modules $\psi^A: \frac{1}{D}\mathcal{V}^A / I_{\mathbf{a}} \frac{1}{D}\mathcal{V}^A \rightarrow V(\mathbf{b})$.

Under the assumption that $b_i \neq b_j + 1$ for $i \neq j$, the $Y(\mathfrak{gl}_N)$ -module $V(\mathbf{b})$ is irreducible, see Proposition 3.6. Since $\psi(v_1^{\otimes n}) = v_1^{\otimes n}$ and $\dim \frac{1}{D}\mathcal{V}^A / I_{\mathbf{a}} \frac{1}{D}\mathcal{V}^A = N^n = \dim V(\mathbf{b})$, the map ψ is an isomorphism of $Y(\mathfrak{gl}_N)$ -modules. \square

The Bethe algebra $\mathcal{B}^{\mathbf{q}}$ preserves the subspaces $(\frac{1}{D}\mathcal{V}^A)_\lambda \subset \frac{1}{D}\mathcal{V}^A$. The image of an element $B_{k,s}^{\mathbf{q}} \in \mathcal{B}^{\mathbf{q}}$ in $\text{End}((\frac{1}{D}\mathcal{V}^A)_\lambda)$ will be denoted by $\tilde{B}_{k,s}^{\mathbf{q},\lambda}$.

The Bethe algebra $\mathcal{B}^{\mathbf{q}=1}$ preserves the subspaces $(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing} \subset \frac{1}{D}\mathcal{V}^A$. Recall the elements $C_{k,s} \in \mathcal{B}^{\mathbf{q}=1}$ introduced by formula (4.4). The image of an element $C_{k,s} \in \mathcal{B}^{\mathbf{q}}$ in $\text{End}((\frac{1}{D}\mathcal{V}^A)_\lambda^{sing})$ will be denoted by $\tilde{C}_{k,s}^{\lambda}$.

Theorem 10.5 ([MTV2, Theorem 3.7]). *The elements $\tilde{C}_{1,1}^\lambda, \dots, \tilde{C}_{N,N}^\lambda$ are scalar operators, and for a variable x , we have*

$$\sum_{k=0}^N (-1)^k \tilde{C}_{k,k}^\lambda \prod_{j=0}^{N-k-1} (x-j) = \prod_{s=1}^N (x - \lambda_s - N + s).$$

□

10.3. Third main result. By formula (10.1), the space $F_k(\frac{1}{D}\mathcal{V}^A)_\lambda$ is one-dimensional if $k = -\sum_{1 \leq i < j \leq N} \lambda_i \lambda_j$. We fix a nonzero element v_λ^A of that space. If λ is such that $\lambda_1 \geq \dots \geq \lambda_N$, then the element v_λ^A lies in the one-dimensional space $F_k(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}$, see (10.2).

The properties of the element v_λ^A were discussed in [RTVZ]. In particular, see there a geometric description of v_λ^A in terms of orbital varieties. The element v_λ^A was denoted in [GRTV] by v_λ^- , see [GRTV, Formula 2.27].

Theorem 10.6. *Assume that $\mathbf{q} \in (\mathbb{C}^\times)^N$ has distinct coordinates. Then*

- (i) *The map $\tilde{\mu}_\lambda^{\mathbf{q}} : F_{k,s}^{\mathbf{q},\lambda} \mapsto \tilde{B}_{k,s}^{\mathbf{q},\lambda}$, $k = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $\tilde{\mu}_\lambda^{\mathbf{q}} : \mathcal{O}_\lambda^{\mathbf{q}} \rightarrow \mathcal{B}^{\mathbf{q}}((\frac{1}{D}\mathcal{V}^A)_\lambda)$ of filtered algebras. The isomorphism $\tilde{\mu}_\lambda^{\mathbf{q}}$ becomes an isomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module $\mathcal{O}_\lambda^{\mathbf{q}}$ and the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^{\mathbf{q}}((\frac{1}{D}\mathcal{V}^A)_\lambda)$ if we identify the algebras $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and $\mathbb{C}[z_1, \dots, z_n]^S$ by the map $\sigma_s \mapsto \sigma_s(\mathbf{z})$, $s = 1, \dots, n$.*
- (ii) *The map $\tilde{\nu}_\lambda^{\mathbf{q}} : \mathcal{O}_\lambda^{\mathbf{q}} \rightarrow (\frac{1}{D}\mathcal{V}^A)_\lambda$, $f \mapsto \tilde{\mu}_\lambda^{\mathbf{q}}(f) v_\lambda^A$, is an isomorphism of filtered vector spaces identifying the $\mathcal{B}^{\mathbf{q}}((\frac{1}{D}\mathcal{V}^A)_\lambda)$ -module $(\frac{1}{D}\mathcal{V}^A)_\lambda$ and the regular representation of $\mathcal{O}_\lambda^{\mathbf{q}}$.*

Proof. The proof of Theorem 10.6 word by word coincides with that of Theorem 5.2. □

Remark. Theorem 10.6 was announced in [GRTV, Theorem 6.4], cf. the remark after Theorem 5.2. As explained in [GRTV], Theorem 6.4 in [GRTV] implies Theorem 6.5 in [GRTV].

Corollary 10.7. *The $\mathcal{B}^{\mathbf{q}}$ -modules $(\mathcal{V}^S)_\lambda$ and $(\frac{1}{D}\mathcal{V}^A)_\lambda$ are isomorphic.*

Proof. The corollary follows from Theorems 5.2 and 10.6. □

10.4. Fourth main result.

Theorem 10.8. *Let λ be a partition on n with at most N parts. Then*

- (i) *The map $\tilde{\mu}_\lambda : G_{k,s} \mapsto \tilde{C}_{k,s}^\lambda$, $k = 1, \dots, N$, $s > 0$, extends uniquely to an isomorphism $\tilde{\mu}_\lambda : \mathcal{O}_\lambda \rightarrow \mathcal{B}^{q=1}((\frac{1}{D}\mathcal{V}^A)_\lambda^{sing})$ of filtered algebras. The isomorphism $\tilde{\mu}_\lambda$ becomes an isomorphism of the $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ -module \mathcal{O}_λ and the $\mathbb{C}[z_1, \dots, z_n]^S$ -module $\mathcal{B}^{q=1}((\frac{1}{D}\mathcal{V}^A)_\lambda^{sing})$ if we identify the algebras $\mathbb{C}[\sigma_1, \dots, \sigma_n]$ and $\mathbb{C}[z_1, \dots, z_n]^S$ by the map $\sigma_s \mapsto \sigma_s(\mathbf{z})$, $s = 1, \dots, n$.*
- (ii) *The map $\tilde{\nu}_\lambda : \mathcal{O}_\lambda \rightarrow (\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}$, $f \mapsto \tilde{\mu}_\lambda(f) v_\lambda^A$, is an isomorphism of filtered vector spaces decreasing the index of filtration by $\sum_{1 \leq i < j \leq N} \lambda_i \lambda_j$. The isomorphism $\tilde{\nu}_\lambda$ identifies the $\mathcal{B}^{q=1}((\frac{1}{D}\mathcal{V}^A)_\lambda^{sing})$ -module $(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}$ and the regular representation of the algebra \mathcal{O}_λ .*

Proof. The proof of Theorem 10.8 word by word coincides with that of Theorem 6.3. □

Corollary 10.9. *The $\mathcal{B}^{q=1}$ -modules $(\mathcal{V}^S)_\lambda^{sing}$ and $(\frac{1}{D}\mathcal{V}^A)_\lambda^{sing}$ are isomorphic.*

Proof. The corollary follows from Theorems 6.3 and 10.8. □

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